

Paths in Hypercubes with Diagonal Steps

Benjamin Alperovich

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Introduction

A classical problem in combinatorics asks how many monotonic paths connect opposite corners of a rectangular grid using only steps to the right (R) and up (U). For instance, to go from $(0, 0)$ to $(6, 5)$ we must make 6 R-moves and 5 U-moves, in some order. The total number of such paths is

$$\binom{11}{5} = \frac{11!}{6!5!} = 462.$$

This question can be generalized by allowing diagonal moves that combine one step right and one step up. From there, the idea extends naturally into three dimensions, and then to an n -dimensional cube. The following sections show the 2D and 3D cases before presenting a general formula.

2D with Diagonal Steps

If we allow diagonal steps (D), where each D represents one R and one U simultaneously:

- With no diagonals: $\binom{11}{5}$.
- With one diagonal: $\frac{10!}{5!4!1!}$.
- With two diagonals: $\frac{9!}{4!3!2!}$.
- With three diagonals: $\frac{8!}{3!2!3!}$.
- With four diagonals: $\frac{7!}{2!1!4!}$.
- With five diagonals: $\frac{6!}{1!5!}$.

Adding these cases gives

$$\frac{11!}{6!5!} + \frac{10!}{5!4!1!} + \frac{9!}{4!3!2!} + \frac{8!}{3!2!3!} + \frac{7!}{2!1!4!} + \frac{6!}{1!5!} = 3653.$$

3D Case

Now consider a cube of side length 4. To go from $(0, 0, 0)$ to $(4, 4, 4)$ using only axis moves (R, U, F), each path has 4 R's, 4 U's, and 4 F's. The number of paths is

$$\frac{12!}{(4!)^3} = 34,650.$$

3D with Space Diagonals

If we also allow space-diagonal steps (D), which replace one R, one U, and one F simultaneously:

- With no diagonals: $\frac{12!}{(4!)^3}$.
- With one diagonal: $\frac{10!}{(3!)^3 1!}$.
- With two diagonals: $\frac{8!}{(2!)^3 2!}$.
- With three diagonals: $\frac{6!}{(1!)^3 3!}$.
- With four diagonals: $\frac{4!}{(0!)^3 4!}$.

Summing these cases gives

$$34,650 + 16,800 + 2,520 + 120 + 1 = 54,091.$$

Observations Leading to the General Formula

Looking at the 2D and 3D cases suggests a pattern:

- The numerator factorial decreases in steps of $n - 1$ as the number of diagonals increases.
- The power on $(x - k)!$ in the denominator equals the dimension n .
- The extra $k!$ in the denominator accounts for rearrangements of the diagonal steps.
- The number of terms in the sum is $x + 1$, corresponding to $k = 0, 1, \dots, x$ possible diagonals.

These observations indicate that the structure of the formula is consistent across dimensions.

Generalization to n Dimensions

In an n -dimensional cube with side length x :

- Each path must move x steps along each axis.
- If k diagonals are taken, each axis requires $(x - k)$ unit steps.
- The total number of steps is $nx - k(n - 1)$.
- The number of sequences is the multinomial coefficient

$$\frac{(nx - k(n - 1))!}{k! (x - k)!^n}.$$

Summing over all possible k gives the general formula:

$$f(n, x) = \sum_{k=0}^x \frac{(nx - k(n - 1))!}{k! (x - k)!^n}.$$

Conclusion

This exploration begins with the basic grid-walking problem and extends to higher dimensions by allowing diagonal moves. The general formula shows that the same reasoning works in any dimension, producing a neat expression for the number of paths. These results connect directly to well-known sequences in 2D (Delannoy numbers) and also match enumerations in 3D and 4D, while emphasizing the simple combinatorial structure behind them.